Frequency domain analysis of linear circuits using synchronous detection

Introduction

In this experiment, we will study the behavior of simple electronic circuits whose response varies as a function of frequency. Our main objective in this lab is to introduce you to synchronous detection—a technique for characterizing systems that exhibit complex response. By complex response, we mean that the response of a system to a mono-frequency drive contains two components: a component that is in-phase with the drive (in-phase component) and a component that is 90° out-of-phase with the drive (quadrature component.)

One way to obtain the frequency response is to drive the system at a single frequency and measure the response of the system at this frequency. The frequency can then be swept to obtain the full frequency response of the system. Lock-in detectors measure the component of the signal that has a definite phase relationship to a reference signal (in most cases, the reference would be the signal used to drive the system.) Lock-in amplifiers have two outputs – an in-phase or X-output, that measures the input signal that is in-phase with the reference, and a quadrature or Y-output, that measures the signal that is 90° out of phase with the reference. In this lab, we will use the Stanford Research Systems SR830 lock-in amplifier. A great introduction to lock-in detection is given in the SR830 manual, which is included at the end of this manual.

As an example of a system that exhibits a frequency dependent response, consider a damped driven harmonic oscillator described by the following differential equation

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = F(t) \tag{1.1}$$

where, $F(t) = a\,e^{-j\omega t}$ is the drive at frequency ω. The steady-state solution to the above equation is given by

$$x(t) = F(t)\left[\frac{(\omega_0^2 - \omega^2) + j\omega\gamma}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}\right]. \tag{1.2}$$

For a real, we have

$$\mathrm{Re}[x(t)] = \frac{a(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}\cos\omega t + \frac{a\gamma\omega}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}\sin\omega t \tag{1.3}$$

Recall, $e^{-j\omega t} = \cos\omega t - j\sin\omega t$ and $j = \sqrt{-1}$. We see that that the response of the system to a pure cosine drive ($\mathrm{Re}[F(t)] = a\cos\omega t$) contains both a cosine term (in-phase with drive) and a sine term (90° out-of-phase.) In general, we can express the system response to a mono-frequency drive at frequency ω in terms of a complex response function $H(\omega) = H_R(\omega) + jH_I(\omega)$.

$$x(t) = H(\omega)F(t) \tag{1.4}$$

For the case of the damped driven harmonic oscillator,

$$\begin{aligned} H(\omega) &= \frac{x(t)}{F(t)} = \frac{(\omega_0^2 - \omega^2) + j\omega\gamma}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2} \\ H_R(\omega) &= \frac{(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2} \\ H_I(\omega) &= \frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2} \end{aligned} \tag{1.5}$$

In practice, $H(\omega)$ is measured by driving the system at a single frequency and comparing the phase of the response relative to the drive. The coefficients of the sine and cosine terms in (1.3) can be obtained by performing the following calculation.

$$\begin{aligned} H_R(\omega) &= \frac{2}{a}\int_0^{2\pi/\omega} \mathrm{Re}[x(t)]\cos\omega t\, dt \\ H_I(\omega) &= \frac{2}{a}\int_0^{2\pi/\omega} \mathrm{Re}[x(t)]\sin\omega t\, dt \end{aligned} \tag{1.6}$$

As we shall see, a lock-in amplifier performs the operations specified by (1.6) to obtain the coefficients of the in-phase and quadrature response as a function of frequency. In so doing, it allows us to find $H_R(\omega)$ and $H_I(\omega)$. A functional representation of a lock-in detector is shown in Fig. 1. All lock-in amplifiers are made up of three basic blocks: (1) a phase-locked-loop (PLL) circuit, (2) mixers that multiply the input signal and the in-phase and quadrature clock signals, and (3) low-pass filters at the output of the mixers. The PLL circuit produces a sinusoidal clock output that has the same frequency and phase as the reference signal. The reference signal is sent to phase-shifter that produc-

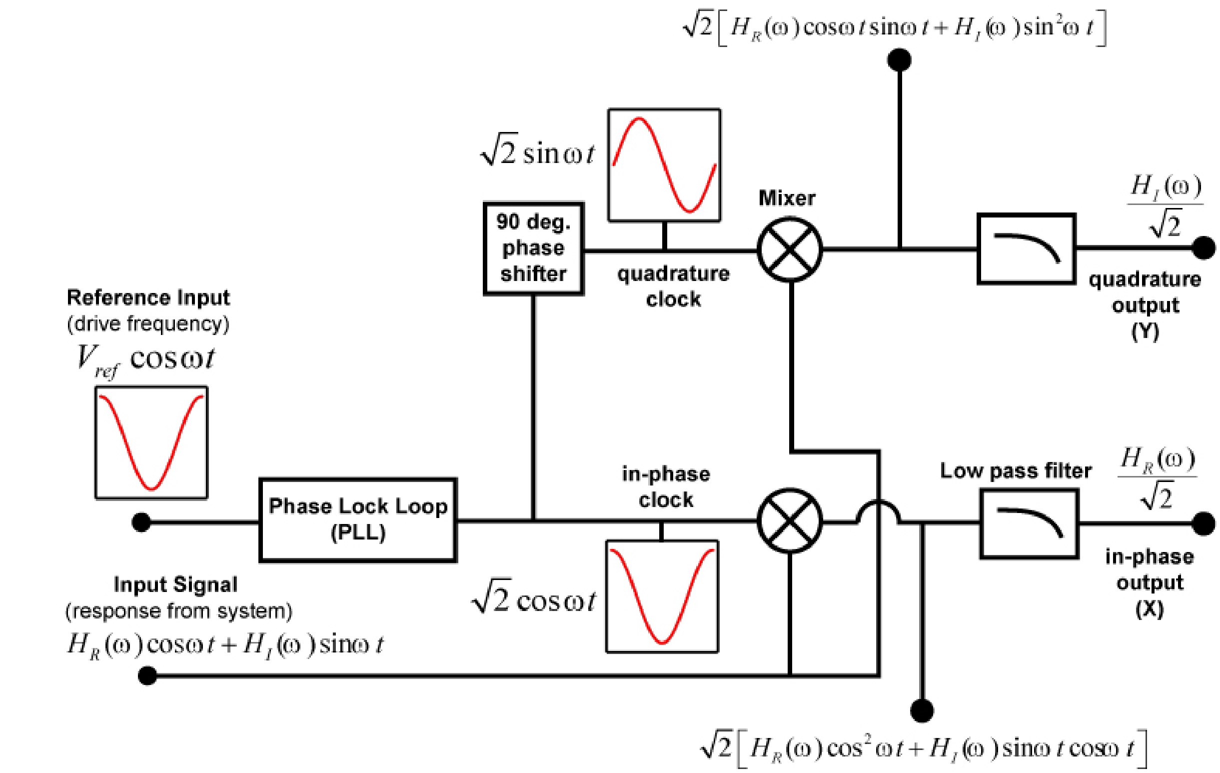

Figure 1: Schematic of lock-in detection.

es a clock signal that is 90° out of phase with respect to the reference. The two mixers multiply each clock signal with the input signal.

The output of the mixers contain a constant term and a term at frequency 2ω.

$$\text{In-phase mixer output:}\quad \sqrt{2}\left[H_R(\omega)\cos^2\omega t + H_I(\omega)\cos\omega t\sin\omega t\right]$$
$$= \frac{1}{\sqrt{2}}\left(1+\cos 2\omega t\right)H_R(\omega) + \frac{1}{\sqrt{2}}\sin 2\omega t\, H_I(\omega)$$
$$(1.7)$$
$$\text{Quadrature mixer output:}\quad \sqrt{2}\left[H_R(\omega)\cos\omega t\sin\omega t + H_I(\omega)\sin^2\omega t\right]$$
$$= \frac{1}{\sqrt{2}}\sin 2\omega t\, H_R(\omega) + \frac{1}{\sqrt{2}}\left(1-\cos 2\omega t\right)H_I(\omega)$$

A low-pass filter is used to remove the high frequency signal at 2ω and only pass the constant term. In this way, the filters perform the same function as integrating over one period.

Frequency Domain Measurement of Complex Impedance

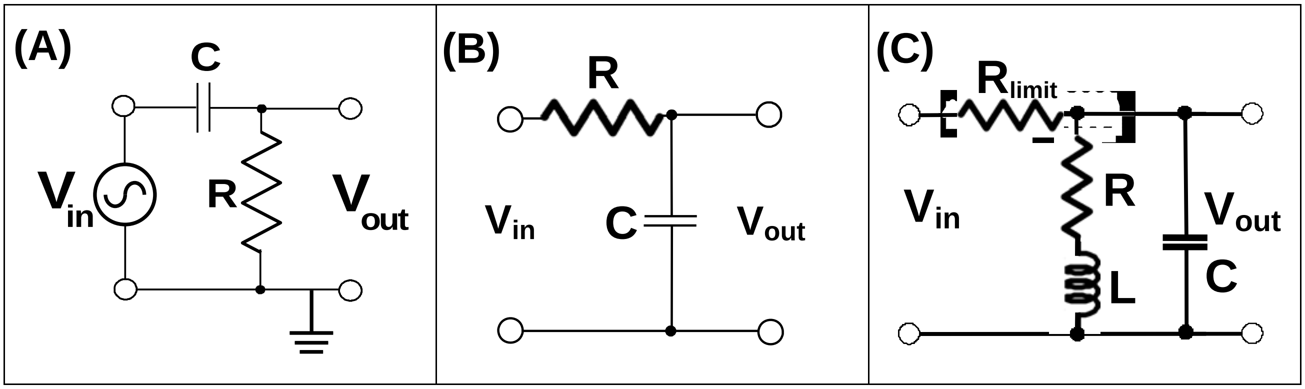

Figure 1: Different RLC circuits for frequency analysis: (A) high-pass filter; (B) low-pass filter; (C) series RLC resonant circuit. In your analysis, it will be helpful to recall the following expressions for the complex impedance for a capacitor and inductor: $Z_c = -j/\omega C$ and $Z_L = j\omega L$.

In this lab, you will analyze the frequency response of the circuits shown in Fig. 1 using lock-in detection.

Let's begin by deriving the response function for each circuit.

$$H(\omega) = \frac{V_{out}}{V_{in}} \tag{1.8}$$

As an example, we'll outline the derivation for the high-pass filter shown in Fig. 1(A). In order to find the voltage across R, we first apply Kirchoff's laws to find the current passing through R.

$$i(t) = \frac{V_{in}}{Z} \tag{1.9}$$

Here, Z is the impedance of the series combination of $Z_c = -j/\omega C$ and R.

$$Z = R - \frac{j}{\omega C} \tag{1.10}$$

and

$$V_{out} = i(t)R = V_{in}\frac{\omega RC}{\omega RC - j} \tag{1.11}$$

thus,

$$
\begin{aligned}
H(\omega) &= \frac{V_{out}}{V_{in}} = \frac{\omega RC}{(\omega RC)^2+1}(\omega RC + j) \\
H_R(\omega) &= \frac{(\omega RC)^2}{(\omega RC)^2+1} \\
H_I(\omega) &= \frac{\omega RC}{(\omega RC)^2+1}
\end{aligned}
\tag{1.12}
$$

Or, in polar form

$$
\begin{aligned}
|H(\omega)| &= \frac{\omega RC}{\sqrt{(\omega RC)^2+1}} \\
\Theta(\omega) &= \tan^{-1}\left(\frac{H_I(\omega)}{H_R(\omega)}\right) = \tan^{-1}\left(\frac{1}{\omega RC}\right)
\end{aligned}
\tag{1.13}
$$

We can see from (1.13) why this circuit is called a high-pass filter. If ω is smaller than a cutoff frequency given by $\omega_c = 1/RC$, then $V_{out}/V_{in} < 1$. Far above this frequency, $V_{out}/V_{in} \to 1$. Thus, the circuit filters out low frequencies and allows high frequencies to pass. In this exercise, we will use a lock-in amplifier to measure both the real and imaginary parts of $H(\omega)$. To see how this is accomplished, it is convenient to represent the input voltage and output voltages as the real part of complex quantities.

$$
\begin{aligned}
V_{in}(t) &= \mathrm{Re}[V_0 e^{-j\omega t}] \\
V_{out}(t) &= \mathrm{Re}[V_0 e^{-j\omega t} H(\omega)]
\end{aligned}
\tag{1.14}
$$

Working through some algebra, we find

$$
\begin{aligned}
V_{in}(t) &= V_0 \cos\omega t \\
V_{out}(t) &= V_0 \frac{\omega RC}{(\omega RC)^2+1}[(\omega RC)\cos\omega t + \sin\omega t]
\end{aligned}
\tag{1.15}
$$

While the input signal is only proportional to $\cos\omega t$, the output contains both $\cos\omega t$ and $\sin\omega t$ terms. The lock-in amplifier has two inputs: 1) a reference signal $V_{ref}(t)$ that has the same frequency and phase of the drive signal $V_{in}(t)$ and 2) the signal to be analyzed $V_{out}(t)$. The in-phase (or X-output) of the lock-in amplifier will thus be proportional to the RMS-amplitude of the $\cos\omega t$ part and the quadrature (or Y output) will be proportional to the RMS-amplitude of the $\sin\omega t$ part.

$$V_X = \frac{V_0}{\sqrt{2}} \frac{(\omega RC)^2}{(\omega RC)^2 + 1}$$
$$V_Y = \frac{V_0}{\sqrt{2}} \frac{\omega RC}{(\omega RC)^2 + 1} \quad (1.16)$$

The factor of $1/\sqrt{2}$ is there because the output of the lock-in amplifier gives root-mean-square (RMS) not peak amplitudes. $H(\omega)$ can be found directly from (1.16).

$$H(\omega) = \frac{\sqrt{2}}{V_0}(V_X + jV_Y) = \frac{\omega RC}{(\omega RC)^2 + 1}(\omega RC + j) \quad (1.17)$$

The response function for parts (B) and (C) are found using the procedure outlined above.

(A) <u>High-pass filter</u>	**(B) <u>Low-pass filter</u>**	**<u>(C) RLC resonant circuit</u>**
$H(\omega) = \frac{\omega RC}{(\omega RC)^2 + 1}(\omega RC + j)$	$H(\omega) = \frac{1 - j\,\omega RC}{(\omega RC)^2 + 1}$	$H(\omega) = \left(\frac{1}{R_{\text{limit}}}\right)\frac{R + j\,\omega L}{1 - \omega^2 LC + j\,\omega RC}$

Table 1: Response functions for the circuits in Fig. 1.

Measurement Procedure

The SR830 comes with a sine-wave function generator output which you will use to drive the circuit. Note that the SR830 function generator output is in V-RMS, so you do not need to include the factor of $\sqrt{2}$ is (1.16) or (1.17). When the lock-in reference is set to "***Internal***," the reference is locked to the sine wave output of the lock-in. V_{out} should be connected to the A-input of the SR830. The sine wave generator of the SR830 has 50 Ω output impedance, thus the magnitude of the impedance of your circuit should be greater than 500 Ω over the frequency range of the measurement in order to ensure that the voltage across the circuit is close to the output voltage of the function generator.

<u>Part I:</u>

For each circuit parts (A) and (B), pick an RC combination that gives a cutoff frequency $f_c = 1/2\pi RC$ between 1 - 5 kHz. Keep the value of R between 0.5-10 kΩ to ensure the circuit impedance is much greater than the output impedance of the SR830 function generator. Place the circuit components on the breadboard, then connect the

V_{in} to the sine wave output of the SR830 and V_{out} to the A-input. For part (C), use a limiting resistor $R_{\text{limit}} = 0.1 M\Omega$, L = 20-50 mH (air filled inductor), C = 1.0 μF, and R = 5 Ω. Drive the circuit with 1.0 V-rms from the function generator. The sensitivity of the lock-in should be set to 1 V/V and the time constant to 300 ms - 12 dB. These values are meant to be guidelines. You are encouraged to experiment with the lock-in settings—in

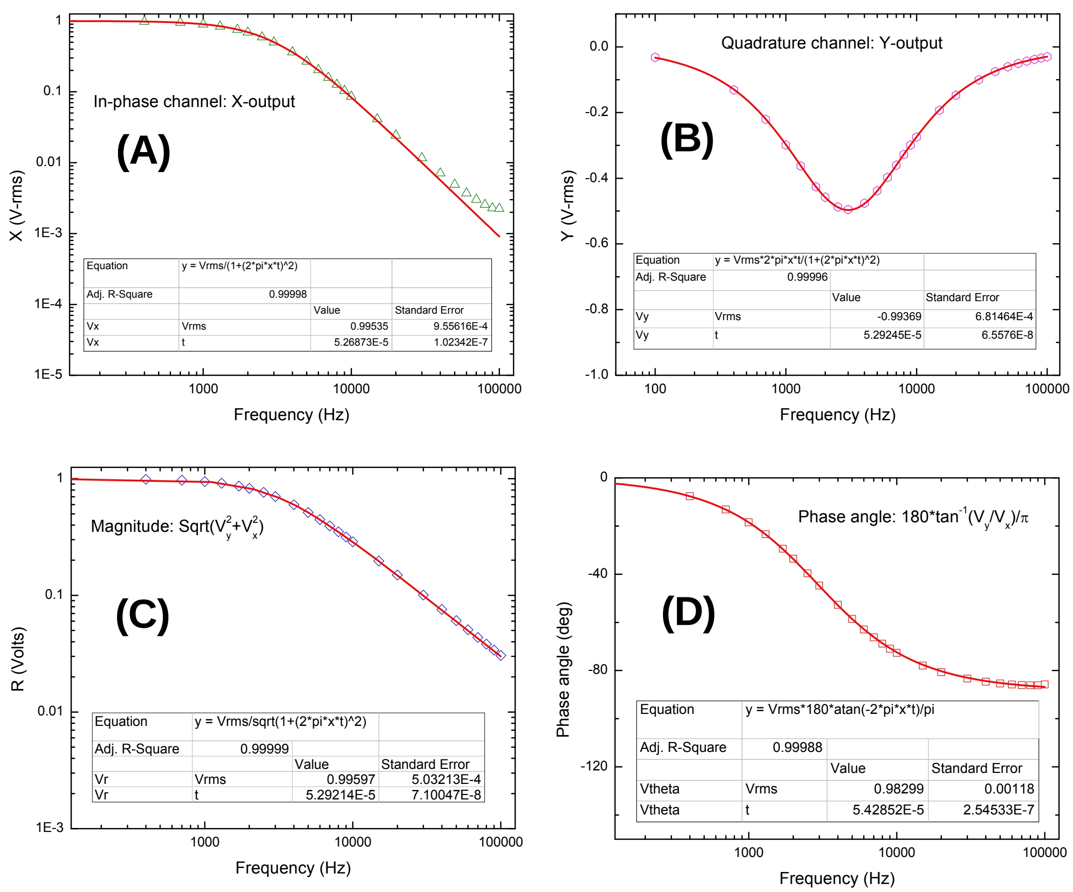

Figure 2: Lock-in measurements taken for a low-pass filter with R = 9950 Ω and C = 5.37 nF. (A) X-output proportional to $H_R(\omega)$, (B) Y-output proportional to $H_I(\omega)$, (C) and (D) are derived from the measured data.

fact, the main point of this exercise is to familiarize you to lock-in detection. The sensitivity sets the gain of the amplifier after the low-pass filters. The time constant sets the cutoff frequency of the low-pass filters. You will notice that the larger the time constant, the more stable the readings appear. However, larger time constants also mean that the output responds more slowly to any changes. When sweeping frequency, it is important to have the time constant of the lock-in to be faster than the sweep rate so that the output can track the changing response. Set the two outputs to X and Y. Both the X and Y output values can be directly read off from the front panel display.

You will measure the real and imaginary parts of the response function by manually sweeping the frequency and recording the X and Y outputs from the lock-in amplifier. Sample data taken for the low-pass filter (Fig. 1(B)) is shown in Fig. 2. Input the data into Origin and make log-log plot for each channel. Use the nonlinear fit option to fit the data to the expected form. Calculate the magnitude and phase of the response function from your data and fit these also using the expected form. All fit functions can be obtained directly from Table 1. Measure the R, C and L using the impedance analyzer and compare to the fitted data.

Part II:

The lock-in amplifier can also be used to measure the response at a harmonic of the drive frequency. This feature is particularly useful when characterizing non-linear systems which create harmonic distortion. As an introduction to this feature of the lock-in, you will use the lock-in amplifier to analyze the harmonic content of a square wave. A periodic waveform can be expressed as a sum of sines and cosines whose frequencies are multiples of the fundamental frequency. The frequency content of the square waveform shown in Fig. 3 can be analyzed by expanding it in terms of a Fourier series.

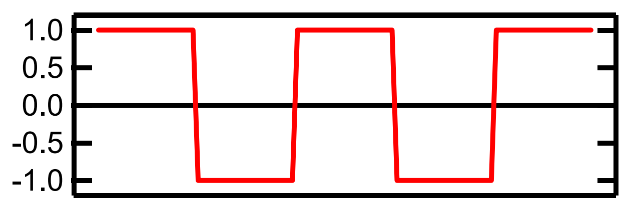

Figure 3: Square waveform

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_0}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T_0}\right) \qquad (1.18)$$

Here, T_0 is the period of the waveform, and the Fourier coefficients are given by

$$
\begin{aligned}
a_n &= \frac{2}{T_0}\int_0^{T_0} F(t)\cos\left(\frac{2\pi nt}{T_0}\right)dt \\
b_n &= \frac{2}{T_0}\int_0^{T_0} F(t)\sin\left(\frac{2\pi nt}{T_0}\right)dt \\
a_0 &= \frac{2}{T_0}\int_0^{T_0} F(t)\ dt
\end{aligned}
\tag{1.19}
$$

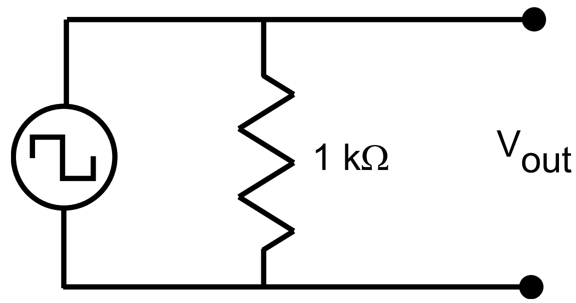

Figure 4: Setup for square waveform measurement.

Use the Wavetek function generator to synthesize the square waveform with a 50% duty cycle. Set the function to bipolar square wave with a peak amplitude of 0.1 V and a frequency of 100 Hz. Connect the function output across a 1 kΩ resistor and connect output voltage to the A-input of the SR830 lock-in amplifier. For this measurement, the reference signal must be supplied by the Wavetek. Connect the ***Snyc*** output of the Wavetek to the reference input of the lock-in and set the reference source to external. Select the harmonic number by pressing the ***Harm*** button. You can select harmonics up to a maximum frequency of 100 kHz. Measure the X and Y output for harmonics up to N=20. Calculate the Fourier coefficients for the square wave analytically using Eq. (1.19) and compare to your measurement.

Report

1. From your data in part I, calculate the real and imaginary parts as well as the magnitude and phase of the response function for the three circuits. Make a log-log plot for each component and fit it using the derived functions given in Table 1.

2. Compare the parameters obtained from your fits to the expected value. Report all errors in the fitted values and compare this error to your uncertainty in the measured values of R, L and C.

3. In part II, plot of amplitudes obtained from the lock-in amplifier vs. 1/n (n is the harmonic number) for a square waveform for the first 20 harmonics. Calculate the Fourier coefficients for the square waveform and compare to your data.